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A NOTE ON ORDERED BELL NUMBERS AND POLYNOMIALS.

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Abstract. In this paper, we study ordered Bell numbers and polynomials and we give some new identities of these numbers and polynomials arising from umbral calculus.

1. Introduction

The ordered Bell numbers are defined by the generating function to be

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},\quad (\text{see } [1,2]). \tag{1.1}$$

Now, we consider the ordered Bell polynomials which are given by the generating function to be

$$\frac{1}{2-e^t}e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\quad (\text{see } [7]). \tag{1.2}$$

Note that

$$\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}=\frac{1}{2-e^t}e^{xt}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}b_{n-l}x^l\right)\frac{t^n}{n!}. \tag{1.3}$$

By (1.3), we get

$$b_n(x)=\sum_{l=0}^{n}\binom{n}{l}b_l x^{n-l}=\sum_{l=0}^{n}\binom{n}{l}b_{n-l}x^l,\quad (n\geq 0). \tag{1.4}$$

Let

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\,\middle|\, a_k\in\mathbb{C}\right\}$$

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be the algebra of formal power series in the variable t with coefficient $\mathbb{C}$.

Suppose that $\mathbb{P}$ be the algebra of polynomials in x over $\mathbb{C}$ and we denote the action of linear functional $L \in \mathbb{P}^*$ on polynomial $p(x)$ by $< L|p(x) >$.

Let $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$. Then we define a linear functional on $\mathbb{P}$ by setting

$$< f(t)|x^n >= a_n, \quad \text{for all } n \geq 0, \quad (\text{see } [3-13]). \tag{1.5}$$

For $L \in \mathbb{P}^*$, there is a unique formal power series $f_L(t)$ such that $L = f_L(t)$ as linear functional on $\mathbb{P}$.

Indeed, if such a formal series $f_L(t)$ exists, then $< L|x^n >=< f_L(t)|x^n >$, and

$$f_L(t) = \sum_{n=0}^{\infty} < f_L(t)|x^n > \frac{t^n}{n!} = \sum_{n=0}^{\infty} < L|x^n > \frac{t^n}{n!}, \quad (\text{see } [11, 13]). \tag{1.6}$$

The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $ord(f(t))$ of $f(t)(\neq 0) \in \mathcal{F}$ is the smallest positive integer k for which the coefficient of the does not vanish.

For $f(t), g(t) \in \mathcal{F}$, we note that

$$< f(t)g(t)|p(x) >=< f(t)|g(t)p(x) >=< 1|f(t)g(t)p(x) >, \quad (\text{see } [13]).$$

Let $f(t), g(t) \in \mathcal{F}$ with $ord(f(t)) = 1$ and $ord(g(t)) = 0$. Then there exists a uniquence sequences $S_n(x)$ such that $< g(t)f(t)^k|S_n(x) >= n!\delta_{n,k}$, $(n, k \geq 0)$, where $\delta_{n,k}$ is the Kronecker's symbol and $S_n(x)$ is a polynomial of degree n. The sequences $S_n(x)$ is called Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$, (see $[7-13]$).

It is well known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} \frac{S_n(x)}{n!} t^n, \quad \text{for all } x \in \mathbb{C}, \tag{1.7}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $f(\bar{f}(t)) = \bar{f}(f(t)) = t$, (see $[6, 13]$). For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} < f(t)|x^k > \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} < t^k|p(x) > \frac{x^k}{k!}. \tag{1.8}$$

Thus, by (1.8), we get

$$p^{(k)}(0) =< t^k|p(x) >=< 1|p^{(k)}(x) >, \tag{1.9}$$

where $p^{(k)}(x) = \left(\frac{d}{dx}\right)^k p(x)$, (see [6 − 9, 13]). By (1.9), we easily get

$$t^k p(x) = p^{(k)}(x),\ e^{yt}p(x) = p(x+y),\ < e^{yt}|p(x) >= p(y). \tag{1.10}$$

In this paper, we give some identities and formulas of ordered Bell polynomials and numbers which are derived from umbral calculus.

2. Ordered Bell numbers and polynomials

From (1.2) and (1.7), we note that

$$b_n(x) \sim (2 - e^t, t),\ (n \geq 0). \tag{2.1}$$

That is, $b_n(x)$ are Appell sequences for $(g(t), t)$. Let

$$\mathbb{P}_n = \left\{p(x) \in \mathbb{C}[x] \,\middle|\, \deg p(x) \leq n\right\}, \quad (n \geq 0).$$

Then $\mathbb{P}_n$ is the $(n+1)$-dimensional vector space. For $S_n(x) \sim (g(t), t)$, we have

$$\frac{1}{g(t)}e^{xt} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.2), we get

$$\begin{aligned} &\frac{1}{g(t)}x^n = S_n(x),\ (n \geq 0). \\ &\Leftrightarrow S_n(x) \sim (g(t), t). \end{aligned} \tag{2.3}$$

Let us take $g(t) = 2 - e^t$, Then we have

$$\sum_{k=0}^{\infty} \frac{b_k(x)}{k!}t^k = \frac{1}{g(t)}e^{xt} = \frac{1}{2-e^t}e^{xt}. \tag{2.4}$$

Thus, we have

$$\frac{1}{2-e^t}x^n = b_n(x),\ (n \geq 0). \tag{2.5}$$

and

$$tb_n(x) = b_n^{'}(x) = nb_{n-1}(x), \quad (n \geq 1). \tag{2.6}$$

Now, we observe that

$$\begin{aligned}\frac{1}{n+1}\Big(b_{n+1}(x+y)-b_{n+1}(x)\Big) &= \frac{1}{n+1}\sum_{k=1}^{\infty}\binom{n+1}{k}b_{n+1-k}(x)y^k\\ &= \sum_{k=1}^{\infty}\frac{n(n-1)\cdots(n-k+2)}{k!}b_{n+1-k}(x)y^k\\ &= \sum_{k=1}^{\infty}\frac{y^k}{k!}t^{k-1}b_n(x).\end{aligned} \tag{2.7}$$

From (2.7), we have

$$\begin{aligned}\int_x^{x+y} b_n(u)du &= \frac{1}{n+1}\Big(b_{n+1}(x+y)-b_{n+1}(x)\Big)\\ &= \sum_{k=1}^{\infty}\frac{1}{k!}y^k t^{k-1}b_n(x)\\ &= \frac{1}{t}\left(\sum_{k=1}^{n}\binom{n}{k}b_{n-k}(x)y^k\right)\\ &= \frac{1}{t}\Big(b_n(x+y)-b_n(x)\Big)\\ &= \frac{1}{t}\big(e^{yt}-1\big)b_n(x),\ (n\geq 0).\end{aligned} \tag{2.8}$$

Thus, by (2.8), we get

$$b_n(x)=\frac{t}{e^t-1}\int_x^{x+1}b_n(u)du=\frac{1}{2-e^t}x^n,\ (n\geq 0). \tag{2.9}$$

Since

$$b_n(x)=t\left(\frac{1}{n+1}b_{n+1}(x)\right),\ (n\geq 0). \tag{2.10}$$

From (2.10), we have

$$\begin{aligned}\Big\langle\frac{e^{yt}-1}{t}\Big|b_n(x)\Big\rangle &= \Big\langle e^{yt}-1\big|\frac{1}{n+1}b_{n+1}(x)\Big\rangle\\ &= \int_0^y b_n(u)du.\end{aligned}$$

For $r\in\mathbb{N}$, the ordered Bell polynomials of order r are defined by the generating function to be

$$\left(\frac{1}{2-e^t}\right)^r e^{xt}=\sum_{n=0}^{\infty}b_n^{(r)}(x)\frac{t^n}{n!}. \tag{2.11}$$

In the special case, $x = 0$, $b_n^{(r)}(0) = b_n^{(r)}$ are called the ordered Bell numbers with order r.

From (1.7) and (2.11), we note that

$$b_n^{(r)}(x) \sim \left((2-e^t)^r, t\right), \ (n \geq 0). \tag{2.12}$$

Thus, by (2.12), we see that $b_n^{(r)}(x)$ is also Appell sequence. It is easy to show that

$$b_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{r} b_{n-l}^{(r)} x^l, \ (n \geq 0), \tag{2.13}$$

and

$$b_n^{(r)} = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, l_2, \cdots, l_r} b_{l_1} b_{l_2} \cdots b_{l_r}, \tag{2.14}$$

where $\binom{n}{l_1, l_2, \cdots, l_r} = \frac{n!}{l_1! l_2! \cdots l_r!}$. From (2.13), we note that

$$\frac{d}{dx} b_n^{(r)}(x) = n b_{n-1}^{(r)}(x), \ (n \in \mathbb{N}). \tag{2.15}$$

By (2.15), we get

$$\begin{aligned} \int_x^{x+y} b_n^{(r)}(u) du &= \frac{1}{n+1} \left\{ b_{n+1}^{(r)}(x+y) - b_{n+1}^{(r)}(x) \right\} \\ &= \sum_{k=1}^{\infty} \frac{y^k}{k!} t^{k-1} b_k(x) = \left(\frac{e^{yt}-1}{t} \right) b_n^{(r)}(x). \end{aligned} \tag{2.16}$$

By (2.11), we easily get

$$-\left(\frac{1}{2-e^t}\right)^r e^{(x+1)t} + 2\left(\frac{1}{2-e^t}\right)^r e^{xt} = \left(\frac{1}{2-e^t}\right)^{r-1} e^{xt}. \tag{2.17}$$

Thus, we have

$$\sum_{n=0}^{\infty} \left(2b_n^{(r)}(x) - b_n^{(r)}(x+1) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} b_n^{(r-1)}(x) \frac{t^n}{n!}. \tag{2.18}$$

By comparing the coefficients on the both sides of (2.18), we get

$$2b_n^{(r)}(x) - b_n^{(r+1)}(x) = b_n^{(r-1)}(x), \ (r \in \mathbb{N}, \ n \geq 0). \tag{2.19}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!} &= \left(\frac{1}{2-e^t}\right)^r e^{xt} = \left(\frac{1}{2-e^t}\right)^{r-1}\left(\frac{1}{2-e^t}\right)e^{xt}\\ &= \left(\sum_{l=0}^{\infty} b_l^{(r-1)}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} b_m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} b_l^{(r-1)}(x)b_{n-l}\right)\frac{t^n}{n!.}\end{aligned} \tag{2.20}$$

Thus, by (2.20), we get

$$b_n^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l} b_l^{(r-1)}(x)b_{n-l},\ (n \geq 0). \tag{2.21}$$

From (2.19) and (2.21), we can derive the following equation:

$$\begin{aligned}\frac{e^t-1}{t} b_n^{(r)}(x) &= \int_x^{x+1} b_n^{(r)}(u)du\\ &= \frac{1}{n+1}\{b_{n+1}^{(r)}(x+1) - b_{n+1}^{(r)}(x)\}\\ &= \frac{1}{n+1}\{b_{n+1}^{(r)}(x+1) - 2b_{n+1}^{(r)}(x) + b_{n+1}^{(r)}(x)\}\\ &= \frac{1}{n+1}\{b_{n+1}^{(r)}(x) - b_{n+1}^{(r-1)}(x)\}\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l} b_l^{(r-1)}(x)b_{n+1-l}.\end{aligned} \tag{2.22}$$

By (2.22), we get

$$\frac{e^t-1}{t} b_n^{(r)}(x) = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n+1}{l}\binom{l}{m} b_{n+1-l}b_{l-m}^{(r)}x^m. \tag{2.23}$$

From (2.23), we note that

$$\begin{aligned}b_n^{(r)}(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n+1}{l}\binom{l}{m} b_{n+1-l}b_{l-m}^{(r)}\frac{t}{e^t-1}x^m\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n+1}{l}\binom{l}{m} b_{n+1-l}b_{l-m}^{(r)}B_m(x),\end{aligned} \tag{2.24}$$

where $B_m(x)$ are ordinary Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Note that

$$\begin{aligned}\frac{e^t-1}{t}b_n^{(r)}(x)&=\frac{1}{n+1}\{b_{n+1}^{(r)}(x)-b_{n+1}^{(r-1)}(x)\}\\&=\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\left(b_l^{(r)}-b_l^{(r-1)}\right)x^{n+1-l}\end{aligned}\tag{2.25}$$

Thus, by (2.25), we get

$$\begin{aligned}b_n^{(r)}(x)&=\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\left(b_l^{(r)}-b_l^{(r-1)}\right)\frac{t}{e^t-1}x^{n+1-l}\\&=\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\left(b_l^{(r)}-b_l^{(r-1)}\right)B_{n+1-l}(x).\end{aligned}\tag{2.26}$$

Since

$$\begin{aligned}\left\langle\frac{e^{yt}-1}{t}\;\middle|\;b_n^{(r)}(x)\right\rangle&=\left\langle e^{yt}-1\;\middle|\;\frac{1}{n+1}b_{n+1}^{(r)}(x)\right\rangle\\&=\int_0^y b_n^{(r)}(u)du.\end{aligned}\tag{2.27}$$

In the special case, $y=1$, we have

$$\begin{aligned}\left\langle\frac{e^{t}-1}{t}\;\middle|\;b_n^{(r)}(x)\right\rangle&=\int_0^1 b_n^{(r)}(u)du\\&=\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}b_l^{(r-1)}b_{n+1-l}.\end{aligned}\tag{2.28}$$

It is not difficult to show that

$$\left\langle\left(\tfrac{1}{2-e^t}\right)^r\;\middle|\;x^n\right\rangle=\sum_{n=l_1+\cdots+l_r}\binom{n}{l_1,\cdots,l_r}\left\langle\tfrac{1}{2-e^t}\;\middle|\;x^{l_1}\right\rangle\times\cdots\times\left\langle\tfrac{1}{2-e^t}\;\middle|\;x^{l_r}\right\rangle,\tag{2.29}$$

and

$$\left\langle\left(\frac{1}{2-e^t}\right)^r\;\middle|\;x^n\right\rangle=b_n^{(r)},\ \text{and}\ \left\langle\left(\frac{1}{2-e^t}\right)\;\middle|\;x^n\right\rangle=b_n,\ (n\geq 0).\tag{2.30}$$

By (2.29) and (2.30), we get

$$\sum_{n=l_1+\cdots+l_r} \binom{n}{l_1,\cdots,l_r} b_{l_1} b_{l_2} \cdots b_{l_r} = b_n^{(r)},\ (n \geq 0). \tag{2.31}$$

Let us take $p(x) \in \mathbb{P}_n,\ (n \geq 0)$, with

$$p(x) = \sum_{k=0}^{n} C_k b_k(x). \tag{2.32}$$

Then, we have

$$\left\langle (2-e^t)t^k \mid b_n(x) \right\rangle = n!\delta_{n,k},\ (n,k \geq 0), \tag{2.33}$$

and

$$\begin{aligned} \left\langle (2-e^t)t^k \mid p(x) \right\rangle &= \sum_{l=0}^{n} C_l \left\langle (2-e^t)t^k \mid b_l(x) \right\rangle \\ &= \sum_{l=0}^{n} C_l \delta_{n,l} l! = k! C_k. \end{aligned} \tag{2.34}$$

Thus, by (2.34), we get

$$C_k = \frac{1}{k!}\left\langle (2-e^t)t^k \mid p(x) \right\rangle = \frac{1}{k!}\left\langle (2-e^t)t^k \mid p^{(k)}(x) \right\rangle. \tag{2.35}$$

From (2.32) and (2.35), we obtain the following equation.

For $p(x) = b_n^{(r)}(x) \in \mathbb{P}_n$, we have

$$p(x) = \sum_{k=0}^{n} C_k b_k(x), \tag{2.36}$$

where

$$\begin{aligned} C_k &= \frac{1}{k!}\left\langle 2-e^t \mid p^{(k)}(x) \right\rangle = \binom{n}{k}\left\langle 2-e^t \mid b_{n-k}^{(r)}(x) \right\rangle \\ &= \binom{n}{k}\left(2b_{n-k}^{(r)} - b_{n-k}^{(r)}(1)\right) = \binom{n}{k} b_{n-k}^{(r-1)}. \end{aligned} \tag{2.37}$$

Hence, by (2.36) and (2.37), we get

$$b_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}^{(r-1)} b_k(x),\ (n \geq 0). \tag{2.38}$$

From $b_n^{(r)}(x) \sim \left((2-e^t)^r, t\right)$, we have

$$\left\langle (2-e^t)^r t^k \mid b_n^{(r)}(x) \right\rangle = n!\delta_{n,k},\ (n,k \geq 0). \tag{2.39}$$

For $p(x) = \sum_{k=0}^{n} C_k^{(r)} b_k^{(r)}(x) \in \mathbb{P}_n$, we have

$$\begin{aligned}\left\langle (2-e^t)^r t^k \mid p(x) \right\rangle &= \sum_{l=0}^{n} C_l^{(r)} \left\langle (2-e^t)^r t^k \mid b_k^{(r)}(x) \right\rangle \\ &= \sum_{l=0}^{n} C_l^{(r)} l! \delta_{l,k} = k! C_k^{(r)}, \ (k \geq 0).\end{aligned} \tag{2.40}$$

Thus, by (2.40), we get

$$C_k^{(r)} = \frac{1}{k!} \left\langle (2-e^t)^r t^k \mid p(x) \right\rangle = \frac{1}{k!} \left\langle (2-e^t)^r \mid p^{(k)}(x) \right\rangle. \tag{2.41}$$

Let us take $p(x) = b_n(x)$ with

$$b_n(x) = p(x) = \sum_{k=0}^{n} C_k^{(r)} b_k^{(r)}(x). \tag{2.42}$$

Then, we have

$$\begin{aligned}C_k^{(r)} &= \frac{1}{k!} \left\langle (2-e^t)^r \mid p^{(k)}(x) \right\rangle = \binom{n}{k} \left\langle (2-e^t)^r \mid b_{n-k}(x) \right\rangle \\ &= \binom{n}{k} \sum_{j=0}^{r} \binom{r}{j} (-1)^j 2^{r-j} \left\langle e^{jt} \middle| b_{n-k}(x) \right\rangle \\ &= \binom{n}{k} \sum_{j=0}^{r} \binom{r}{j} (-1)^j 2^{r-j} b_{n-k}(j).\end{aligned} \tag{2.43}$$

By (2.42) and (2.43), we get

$$b_n(x) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r} \binom{n}{k} \binom{r}{j} (-1)^j 2^{r-j} b_{n-k}(j) \right) b_k^{(r)}(x). \tag{2.44}$$

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